

WEIGHTED ITERATES AND VARIANTS OF THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

BY

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ABSTRACT. In a recent paper, M. A. Leckband and C. J. Neugebauer obtained a rearrangement inequality for a generalized maximal operator with respect to two measures. For an application they studied norm bounds for the iterated Hardy-Littlewood maximal operator with respect to two measures. In this paper this theory is further developed and other applications of the rearrangement inequality are obtained.

1. Let μ, ν be two measures on $\mathbf{R}^n$, and let there be associated with each cube $Q \subset \mathbf{R}^n$ a function ϕ_Q supported in Q . We consider the maximal operator

$$Mf(x) = \sup \int f \phi_Q d\nu$$

where the sup is extended over all Q with center x . If g_λ^* is the nonincreasing rearrangement of g with respect to the measure λ , i.e., $g_\lambda^*(t) = \inf\{y: \lambda\{|g| > y\} \leq t\}$, and if $\Phi(t) = \sup_Q \{\mu(Q) \phi_{Q,\nu}^*(\mu(Q)t)\}$, then we have proved, in [6], the following theorem.

THEOREM 1. $(Mf)_\mu^*(\xi) \leq A \int_0^\infty \Phi(t) f_\nu^*(t\xi) dt$.

From this rearrangement inequality it is easy to get general norm inequalities. In particular, Minkowski's integral inequality gives

$$\|Mf\|_{p,\mu} \leq A \left(\int_0^\infty \frac{\Phi(t)}{t^{1/p}} dt \right) \|f\|_{p,\nu},$$

and thus, if $\Phi \in L(p', 1)$ (see [2]) we get a weighted norm inequality. At this point Muckenhoupt's A_p -condition enters; thus, if $(u, v) \in A_p$, i.e., $\int_Q u (f_Q v^{1-p'})^{p-1} \leq C |Q|^p$ [7], and if $d\mu = u dx$, $d\nu = v dx$, $\phi_Q(x) = (1/|Q|)(\chi_Q(x)/v(x))$, then the above $Mf(x) = \sup (1/|Q|) \int_Q f(t) dt$, the familiar Hardy-Littlewood maximal function. We have proved in [6] that if $(u, v) \in A_p$, $1 < p < \infty$, then $\Phi_{u,v} \in L(p', \infty)$, and $u = v$ is in A_p , $1 < p < \infty$, if and only if $\Phi_u \in L(p', 1)$, where

$$\Phi_{u,v}(t) = \sup_Q \left\{ \frac{u(Q)}{|Q|} \left(\frac{\chi_Q}{v} \right)_\nu^* (\mu(Q)t) \right\}$$

and $\Phi_u = \Phi_{u,u}$.

Received by the editors February 26, 1982.

1980 *Mathematics Subject Classification*. Primary 42B25.

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0002-9947/82/0000-1126/\$03.25

The j th iterated Hardy-Littlewood maximal function $M_j f$ turns out to be crucial in the extrapolation problem, i.e., when does $\|Mf\|_{p,u} \leq A_p \|f\|_{p,v}$ imply the existence of $\varepsilon > 0$ so that $\|Mf\|_{p-\varepsilon,u} \leq B \|f\|_{p-\varepsilon,v}$? We have shown in [6] that extrapolation is possible if $\|M_j\| = \mathcal{O}(A^j)$ as $j \rightarrow \infty$, where $\|M_j\|$ is the norm of M_j as an operator from L_u^p to L_v^p .

Using a dense set of functions, namely those which are nowhere constant, we obtain (Lemma 2)

$$M_{j+1}f(x) \leq B_n^j \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \frac{\log^j(|Q|/\rho_Q(y))}{j!} dy,$$

where

$$\rho_Q(y) = \inf \left\{ t : y \in \left\{ x \in Q : |f(x)| \geq (f\chi_Q)^*(t) \right\} \right\}.$$

From this we obtain (Theorem 2) that if $(u, v) \in A_p$, then for each $q > p$ there is a constant $0 < A_q < \infty$ such that $\|M_j f\|_{q,u} \leq A_q^j \|f\|_{q,v}$. This then implies that extrapolation of $\|Mf\|_{p,u} \leq B \|f\|_{p,v}$ is possible if and only if $\|M_j\| = \mathcal{O}(A^j)$ as $j \rightarrow \infty$ (Theorem 3).

At a fixed p , iteration may not be possible, i.e., if $\|M_j f\|_{p,u} \leq A \|f\|_{p,v}$ for some $j > 0$, then M_{j+1} may not be bounded on L_v^p . We will also study conditions under which iterations in this case are possible by estimating the associated $\Phi(t)$ (Theorem 4).

The inequality in Theorem 1 readily lends itself to studying restricted weak type. In particular, $\Phi \in L(p', \infty)$ shows that Mf is restricted weak type (p, p) , i.e., $\|Mf\|_{p,\infty,\mu} \leq A \|f\|_{p,1,\nu}$. This observation is used to give a simple proof of the weak type behavior of a generalization of a maximal operator recently studied by Stein [9] (Theorem 6). For the usual Hardy-Littlewood maximal operator we will see (Theorem 7) that $\|Mf\|_{p,\infty,u} \leq C \|f\|_{p,1,v}$, $1 < p < \infty$, if and only if $\Phi \in L(p', \infty)$. We believe that this characterization is easier to use than the one found by Kerman [5]. The paper concludes with some variants of Theorem 1.

2. We will establish an inequality similar to Theorem 1 for the j th iterated Hardy-Littlewood maximal operator $M_j f$. It will be convenient to define a “telescoping” maximal operator $\bar{M}_j f$ as follows. First, define

$$\bar{M}_{1Q}f(x) = \sup_{x \in Q_1 \subset Q} \frac{1}{|Q_1|} \int_{Q_1} |f|, \quad \bar{M}_{jQ}f(x) = \sup_{x \in Q_j \subset Q} \frac{1}{|Q_j|} \int_{Q_j} \bar{M}_{j-1,Q_j}f,$$

i.e.,

$$\begin{aligned} \bar{M}_{jQ}f(x) = & \sup_{x \in Q_1 \subset Q} \frac{1}{|Q_1|} \int_{Q_1} \sup_{x_2 \in Q_2 \subset Q_1} \frac{1}{|Q_2|} \int_{Q_2} \\ & \cdots \frac{1}{|Q_{j-1}|} \int_{Q_{j-1}} \sup_{x_j \in Q_j \subset Q_{j-1}} \frac{1}{|Q_j|} \int_{Q_j} |f(t)| dt dx_j \cdots dx_2. \end{aligned}$$

Let $\bar{M}_1 f = Mf$, and for $j \geq 2$, define

$$\bar{M}_j f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q \bar{M}_{j-1,Q}f(t) dt.$$

LEMMA 1. Let $f \in L^p(\mathbf{R}^n)$ for some $p > 1$. Then there is $C_n > 0$ such that for a.e. x ,

$$M_j f(x) \leq C_n^j \bar{M}_j f(x), \quad j = 1, 2, \dots$$

PROOF. The condition $p > 1$ assures that for a.e. x , $M_j f(x) < \infty$, $j = 1, 2, \dots$, and we will show that the lemma holds for all such x 's. Assume the inequality is true for $j - 1$ and $M_j f(x) < \infty$. Then

$$\begin{aligned} M_j f(x) &= M(M_{j-1} f)(x) \leq C_n^{j-1} M(\bar{M}_{j-1} f)(x) \\ &< C_n^{j-1} \left(\frac{1}{|Q|} \int_Q \bar{M}_{j-1} f(u) du + \varepsilon \right) \end{aligned}$$

for some Q containing x . We now let

$$S = \left\{ u \in Q : \frac{1}{2} M(\bar{M}_{j-1} f)(x) \leq \bar{M}_{j-1} f(u) < \infty \right\},$$

and we note that

$$M(\bar{M}_{j-1} f)(x) \leq \frac{2}{|Q|} \int_S \bar{M}_{j-1} f(u) du + 2\varepsilon.$$

For each $u \in S$ choose a cube Q_u with $u \in Q_u$ and

$$\bar{M}_{j-1} f(u) \leq \frac{1}{|Q_u|} \int_{Q_u} \bar{M}_{j-2, Q_u} f(t) dt + \varepsilon.$$

Since $\sup_{u \in S} \{|Q_u|\} < \infty$, select $u_0 \in S$ for which $l_{u_0} \geq \frac{1}{2} \sup_{u \in S} l_u$, where l_u is the sidelength of Q_u .

There are now two cases: $3Q \supset Q_{u_0}$ and $3Q_{u_0} \supset Q$. Then $5Q \supset Q_u$ and $5Q_{u_0} \supset Q_u$ for $u \in S$, respectively. Thus, in the first case,

$$M(\bar{M}_{j-1} f)(x) \leq \frac{2 \cdot 5^n}{|5Q|} \int_S \sup_{t \in Q_1 \subset 5Q} \frac{1}{|Q_1|} \int_{Q_1} \bar{M}_{j-2, Q_1} f(y) dy dt + 2\varepsilon$$

and, hence,

$$M_j f(x) \leq \frac{2 \cdot 5^n C_n^{j-1}}{|5Q|} \int_{5Q} \bar{M}_{j-1, 5Q} f + 2\varepsilon \leq C_n^j \bar{M}_j f(x) + 2\varepsilon.$$

In the second case we get for $z \in 5Q_{u_0}$,

$$\varepsilon + \sup_{z \in Q_1 \subset 5Q_{u_0}} \frac{5^n}{|Q_1|} \int_{Q_1} \bar{M}_{j-2, Q_1} f \geq \varepsilon + \frac{5^n}{|5Q_{u_0}|} \int_{Q_{u_0}} \bar{M}_{j-2, Q_{u_0}} f \geq \frac{1}{2} M(\bar{M}_{j-1} f)(x),$$

and thus,

$$\begin{aligned} M(\bar{M}_{j-1} f)(x) &\leq 2\varepsilon + \frac{2 \cdot 5^n}{|5Q_{u_0}|} \int_{5Q_{u_0}} \sup_{z \in Q_1 \subset 5Q_{u_0}} \frac{1}{|Q_1|} \int_{Q_1} \bar{M}_{j-2, Q_1} f \\ &= 2\varepsilon + \frac{C_n}{|5Q_{u_0}|} \int_{5Q_{u_0}} \bar{M}_{j-1, 5Q_{u_0}} f \leq 2\varepsilon + C_n \bar{M}_j f(x). \end{aligned}$$

This completes the proof.

For $f: \mathbf{R}^n \rightarrow [0, \infty]$, let f^* be the rearrangement of f relative to Lebesgue measure on $\mathbf{R}^n$. We will also assume that f is nowhere constant, i.e., $|\{x: f(x) = a\}| = 0$,

$a > 0$. For a cube $Q \subset \mathbf{R}^n$, let $E_t = \{x \in Q: f(x) \geq (f \cdot \chi_Q)^*(t)\}$. Then $|E_t| = t$ and $t^{-1} \int_0^t (f \chi_Q)^* = t^{-1} \int_{E_t} f \chi_Q$.

LEMMA 2. Let $f: \mathbf{R}^n \rightarrow [0, \infty]$ be in $L^p(\mathbf{R}^n)$ for some $p > 1$ and nowhere constant. Then there exists $B_n > 0$ such that for a.e. x ,

$$M_{j+1} f(x) \leq B_n^j \sup_{x \in Q} \frac{1}{|Q|} \int_Q f(y) \frac{\log^j(|Q|/\rho_Q(y))}{j!} dy,$$

where $\rho_Q(y) = \inf\{t: y \in E_t\}$.

PROOF. By Lemma 1, for a.e. x , $M_{j+1} f(x) \leq C_n^{j+1} \bar{M}_{j+1} f(x)$. By the n -dimensional version of Lemma 1 of [6] we see that

$$\bar{M}_{j+1} f(x) \leq A_n^j \sup_{x \in Q} \frac{1}{|Q|} \int_0^{|Q|} \frac{1}{t_1} \int_0^{t_1} \cdots \frac{1}{t_j} \int_0^{t_j} (f \cdot \chi_Q)^*(\tau) d\tau dt_j \cdots dt_1.$$

For the proof of the lemma we assume, for simplicity $j = 2$. Then

$$\begin{aligned} & \frac{1}{|Q|} \int_0^{|Q|} \frac{1}{t} \int_0^t \frac{1}{s} \int_0^s (f \cdot \chi_Q)^*(u) du ds dt \\ &= \frac{1}{|Q|} \int_0^{|Q|} \frac{1}{t} \int_0^t \frac{1}{|E_s|} \int_{E_s} f(x) dx ds dt = \frac{1}{|Q|} \int_0^{|Q|} \frac{1}{t} \int_{E_t} f(x) \int_{\rho_Q(x)}^t \frac{1}{s} ds dx dt \\ &= \frac{1}{|Q|} \int_0^{|Q|} \frac{1}{t} \int_{E_t} f(x) \log\left(\frac{t}{\rho_Q(x)}\right) dx dt = \frac{1}{|Q|} \int_Q f(x) \frac{1}{2} \log^2\left(\frac{|Q|}{\rho_Q(x)}\right) dx. \end{aligned}$$

Now let (u, v) be a pair of weights with $u \geq 0$ in $L^1_{\text{loc}}(\mathbf{R}^n)$ and $0 < v < \infty$, a.e. Set $d\mu = u dx$, $d\nu = v dx$.

LEMMA 3. With the same hypothesis as in Lemma 2,

$$(M_{j+1} f)_\mu^*(\xi) \leq A B_n^j \int_0^\infty \Phi_j(t) f_\nu^*(t\xi) dt,$$

where

$$\Phi_j(t) = \Phi_{j,f}(t) = \sup_Q \left\{ \frac{\mu(Q)}{|Q|} \left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \chi_Q(x) \right)_\nu^* (\mu(Q)t) \right\}.$$

PROOF. This is Theorem 1 with

$$\phi_Q(x) = \frac{1}{|Q|} \frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \chi_Q(x)$$

coupled with Lemma 2.

THEOREM 2. Let (u, v) be a pair of weights as above and assume that $\|Mf\|_{q,u} \leq B_q \|f\|_{q,v}$, $1 \leq p < q$. Then for each $q > p$ there is a constant $0 < A_q < \infty$ such that $\|M_j f\|_{q,u} \leq A_q^j \|f\|_{q,v}$.

PROOF. We may assume that f is nowhere constant. We will estimate $\Phi_j(t)$ of Lemma 3 and show that $\Phi_j \in L(q', \infty)$, $q > p$. Fix $q > p$, and let $p < p_0 < q$. Since

$\|Mf\|_{p_0, u} \leq B_{p_0} \|f\|_{p_0, v}$, the pair $(u, v) \in A_{p_0}$, i.e.,

$$\mu(Q) \left(\int_Q v^{1-p'_0} \right)^{p_0-1} \leq C |Q|^{p_0} \quad [7].$$

We next note that

$$\begin{aligned} \left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \chi_Q(x) \right)_v^* (t) &\leq \frac{1}{t^{1/q'}} \left(\int_Q \frac{\log^{jq'}(|Q|/\rho_Q(x))}{j!^{q'}} v^{1-q'} dx \right)^{1/q'} \\ &\leq \frac{1}{j!t^{1/q'}} \left(\int_Q \log^{jq'r'} \left(\frac{|Q|}{\rho_Q(x)} \right) dx \right)^{1/r'q'} \left(\int_Q v^{1-p'_0} \right)^{1/q(p'_0-1)} \end{aligned}$$

where $r = (p'_0 - 1)/(q' - 1) > 1$. Since the rearrangement of $\log |Q|/\rho_Q(x)$ is $\log |Q|/t$, and $(1/a) \int_0^a \log^r(a/t) dt = \Gamma(r+1)$, $r > -1$, the above equals

$$\frac{\Gamma(jq'r' + 1)^{1/r'q'}}{t^{1/q'} j!} |Q|^{1/r'q'} \left(\int_Q v^{1-p'_0} \right)^{1/q(p'_0-1)}.$$

Let $c_j = \Gamma(jq'r' + 1)^{1/r'q'}/j!$ and observe that from Stirling's formula one gets $C_j^{1/j} \leq C_*$, where C_* is a constant depending on the product $q'r'$ only. All this gives

$$\begin{aligned} \frac{\mu(Q)}{|Q|} \left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \chi_Q(x) \right)_v^* (\mu(Q)t) \\ \leq \frac{C_*^j}{t^{1/q'}} \cdot \frac{\mu(Q)^{1/q}}{|Q|^{1-1/r'q'}} \left(\int_Q v^{1-p'_0} \right)^{1/q(p'_0-1)}. \end{aligned}$$

We finally observe that $1 - 1/r'q' = p_0/q$, $(p_0 - 1)(p'_0 - 1) = 1$, and thus $\Phi_j(t) \leq CC_*^j/t^{1/q'}$.

We now complete the proof and fix $s > p$. Then from Lemma 3 we get

$$\|M_{j+1}f\|_{s, u} \leq AB_n^j \left(\int_0^\infty \frac{\Phi_j(t)}{t^{1/s}} dt \right) \|f\|_{s, v}.$$

Now choose $p < q_1 < s$ and $s < q_2 < \infty$ and observe that

$$\int_0^\infty \frac{\Phi_j(t)}{t^{1/s}} dt = \int_0^1 + \int_1^\infty \leq \int_0^1 \frac{\alpha^j}{t^{1/q_1+1/s}} dt + \int_1^\infty \frac{\beta^j}{t^{1/q_2+1/s}} dt \leq \gamma^j,$$

and the proof of Theorem 2 is complete.

3. In [6] we have shown that if $\|M_j f\|_{p, u} \leq A_j \|f\|_{p, v}$ for some $1 < p, j = 1, 2, \dots$, and if

$$\Phi(t) = \sup_Q \left\{ \frac{\mu(Q)}{|Q|} \left(\frac{\chi_Q}{v} \right)_v^* (\mu(Q)t) \right\},$$

then

$$\Phi(2^{-N}) \leq C \frac{A_{j+1}}{B^j} \left(\frac{j!}{N^j} \right) 2^{N/p'}.$$

From this and $A_j = \mathcal{C}(A^j)$, one gets that $\Phi \in L((p - \varepsilon)', 1)$ for some $\varepsilon > 0$, and thus $\|Mf\|_{p-\varepsilon, u} \leq A\|f\|_{p-\varepsilon, v}$. All this was done in the context $n = 1$, which we shall also assume for the next theorem.

THEOREM 3. *Let $(u, v) \in A_p$ for some $p > 1$. Then there is $\varepsilon > 0$ with $(u, v) \in A_{p-\varepsilon}$ if and only if $\sup_{\|f\|_{p, v} = 1} \|M_j f\|_{p, u} = \mathcal{C}(A^j)$.*

PROOF. If $(u, v) \in A_{p-\varepsilon}$, the result follows from Theorem 2, and the converse was just mentioned and is Theorem 6 in [6].

There are examples which show that the norm inequality $\|Mf\|_{p, u} \leq A\|f\|_{p, v}$ does not admit an iteration (e.g. Theorem 4 in [6]).

If we let

$$\Phi_0(t) = \sup \left\{ \frac{\mu(Q)}{|Q|} \left(\frac{\chi_Q}{v} \right)_v^* (\mu(Q)t) \right\},$$

the next theorem gives an estimate of $\Phi_j(t)$ in terms of $\Phi_0(t)$ that may allow an iteration up to a certain index.

THEOREM 4. *Let $f: \mathbf{R}^n \rightarrow [0, \infty]$ be in $L^p(\mathbf{R}^n)$ for some $p > 1$ and nowhere constant. Then*

$$\Phi_j(2^{-N}) \leq C \left[\frac{N^j}{j!} \Phi_0 \left(\frac{2^{-N}}{2} \right) + N^j \right], \quad N = 1, 2, \dots$$

PROOF. For a fixed N consider

$$L_{Q, j} = \frac{\mu(Q)}{|Q|} \left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \chi_Q(x) \right)_v^* (\mu(Q)2^{-N}).$$

Let $Q_N = \{x \in Q: \log(|Q|/\rho_Q(x)) > N/\log_2 e\}$ or $Q_N = \{x \in Q: \rho_Q(x) < |Q|2^{-N}\}$. We note that $|Q_N| = |Q|2^{-N}$ and thus

$$\begin{aligned} L_{Q, j} &\leq C \frac{\mu(Q)}{|Q|} \left(\frac{N^j}{j!v(x)} \cdot \chi_{Q \setminus Q_N}(x) \right)_v^* \left(\mu(Q) \frac{2^{-N}}{2} \right) \\ &\quad + \frac{\mu(Q)}{|Q|} \left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \chi_{Q_N}(x) \right)_v^* \left(\mu(Q) \frac{2^{-N}}{2} \right). \end{aligned}$$

The first expression on the right is at most $(N^j/j!)\Phi_0(2^{-N-1})$. The second term is zero if $v(Q_N) \leq \mu(Q)2^{-N-1}$. Hence we assume that $v(Q_N) > \mu(Q)2^{-N-1}$ and construct a set $S_N \subset Q_N$ such that if we let

$$\alpha_N = \left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \chi_{Q_N}(x) \right)_v^* (\mu(Q)2^{-N-1}),$$

then

(i)

$$\frac{1}{2}\mu(Q)2^{-N-1} \leq v(S_N) \leq \mu(Q)2^{-N-1},$$

(ii)

$$\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \geq \alpha_N \quad \text{for } x \in S_N.$$

From this we get

$$\begin{aligned} \alpha_N \nu(S_N) &\leq \int_{Q_N} \frac{\log^j(|Q|/\rho_Q(x))}{j!} dx \leq \int_0^{|Q_N|} \frac{\log^j(|Q|/t)}{j!} dt \\ &= |Q_N| \sum_{k=0}^j \frac{\log^k(|Q|/|Q_N|)}{k!} \leq C |Q_N| \log^j \left(\frac{|Q|}{|Q_N|} \right). \end{aligned}$$

Thus

$$\alpha_N \leq C \frac{|Q_N| \log^j(|Q|/|Q_N|)}{\nu(S_N)} \quad \text{and} \quad \frac{\mu(Q)}{|Q|} \alpha_N \leq CN^j,$$

and the proof is complete.

COROLLARY. Let $(u, v) \in A_p$ for some $1 < p < \infty$. If $\Phi_0(2^{-N}) \leq (C/N^k)2^{N/p'}$, $N = 1, 2, \dots$, then $\|M_j f\|_{p,u} \leq A_j \|f\|_{p,v}$, $j = 1, 2, \dots, k-1$, and

$$\mu\{x: M_k f(x) > y\} \leq (C/y^p) \|f\|_{p,v}^p.$$

PROOF. From Lemma 3 and Minkowski's integral inequality we get

$$\|M_{k-1}\|_{p,u} \leq C \left(\int_0^\infty \frac{\Phi_{k-2}(t)}{t^{1/p}} dt \right) \|f\|_{p,v}.$$

We write $f_0^\infty = f_0^1 + f_1^\infty$, and to estimate f_0^1 we use Theorem 4 and note that

$$\Phi_{k-2}(2^{-N}) \leq C \left(\frac{2^{N/p'}}{N^2} + N^{k-2} \right).$$

From this, $\sum \Phi_{k-2}(2^{-N})/2^{N/p'} < \infty$ and $\Phi_{k-2} \in L(p', 1)$ on $[0, 1]$.

For the integral f_1^∞ we use the proof in Theorem 2, where it was shown that $\Phi_{k-2} \in L(q', \infty)$, $q > p$. Thus

$$\int_1^\infty \frac{\Phi_{k-2}(t)}{t^{1/p}} dt \leq C \int_1^\infty \frac{dt}{t^{1/q' + 1/p}} < \infty.$$

The weak type estimate follows again from Lemma 3 by noting that by Hölder's inequality,

$$(M_k f)_\mu^*(\xi) \leq C \|\Phi_{k-1}\|_{p'} \cdot \|f\|_{p,v} \cdot 1/\xi^{1/p}.$$

Finally,

$$\|\Phi_{k-1}\|_{p'}^{p'} = \int_0^\infty \Phi_{k-1}^{p'} dt = \int_0^1 + \int_1^\infty.$$

Now

$$\int_0^1 \Phi_{k-1}^{p'} dt \leq C \sum \Phi_{k-1}^{p'}(2^{-N}) 2^{-N},$$

and, from Theorem 4, $\Phi_{k-1}^{p'}(2^{-N}) \leq C(2^N/N^{p'} + N^{p'(k-1)})$. Hence $\int_0^1 < \infty$. As before, $\int_1^\infty \leq C \int_1^\infty dt/t^{p'/q'} < \infty$, since $q > p$.

REMARK. The above corollary can be viewed as the converse of Theorem 5 in [6].

4. In this section we will show how Theorem 5 of [6] can be used to obtain extrapolation results from the magnitude of $\|M_j\|$. We assume $n = 1$, the setting in which Theorem 5 of [6] has been proved.

From Theorem 2 we have that if $\|Mf\|_{q,u} \leq B_q \|f\|_{q,v}$, $q > p > 1$, then $\sup_{\|f\|_{q,v}=1} \|M_j f\|_{q,u} = \mathcal{O}(A_q^j)$.

THEOREM 5. Let $1 < p < \infty$. There is a constant $C_p > 0$ such that the following holds. If for some $r_0 > p$, $A_{r_0} \leq C_p/(r_0 - p)$, then $\|M_j f\|_{p,u} \leq B_j \|f\|_{p,v}$, $j = 1, 2, \dots$

PROOF. From Theorem 5 in [6] we have constants $C > 0$, $B > 0$ so that for $r > p$,

$$\begin{aligned} \Phi_0(2^{-N}) &\leq CA_r (BA_r)^j \left(\frac{j!}{N^j} \right) 2^{N(1/p-1/r)} \cdot 2^{N/p'} \\ &\leq CA_r \left(\frac{BA_r \cdot j}{eN} \right)^j j^{1/2} 2^{N(1/p-1/r)} \cdot 2^{N/p'} \end{aligned}$$

by Stirling's formula. Hence, if $\alpha_r = e/2BA_r$, $j = [\alpha_r N]$, we get

$$\Phi_0(2^{-N}) \leq CA_r \left(\frac{1}{2}\right)^{\alpha_r N} (\alpha_r N)^{1/2} 2^{N(1/p-1/r)} \cdot 2^{N/p'}.$$

Hence, from Theorem 4,

$$\Phi_k(2^{-N}) \leq C \left(\frac{N^k}{k!} A_r \left(\frac{1}{2} \right)^{\alpha_r N} (\alpha_r N)^{1/2} 2^{N(1/p-1/r)} \cdot 2^{N/p'} + N^k \right).$$

Now let $C_p = ep^2/2B$, and let $r_0 > p$, for which $(r_0 - p)A_{r_0} \leq C_p$. Then

$$\alpha_{r_0} \geq \frac{r_0 - p}{p^2} > \frac{r_0 - p}{r_0 p} = \frac{1}{p} - \frac{1}{r_0}.$$

We claim now that

$$A_{r_0}(\alpha_{r_0} N)^{1/2} 2^{N(1/p-1/r_0)} \leq N^{-k-2} 2^{\alpha_{r_0} N}, \quad N \geq N_0.$$

With log to the base 2 this is

$$\frac{1}{N} \log A_{r_0} + \frac{1}{N} \log \left(\sqrt{\alpha_{r_0} N} N^{2+k} \right) + \left(\frac{1}{p} - \frac{1}{r_0} \right) \leq \alpha_{r_0}.$$

Since $\alpha_{r_0} > 1/p - 1/r_0$, this is possible for $N \geq N_0$. Hence, $\sum \Phi_k(2^{-N})/2^{N/p'} < \infty$ and $\Phi_k \in L(p', 1)$ on $[0, 1]$. From Theorem 2, $\Phi_k \in L(q', \infty)$, $q > p$, from which $\Phi_k \in L(p', 1)$ on $[1, \infty)$. Minkowski's integral inequality applied to Lemma 3 now completes the proof.

5. We will show in this section how Theorem 1 can be used to study the restricted weak type behavior of a general maximal operator. From Theorem 1 one obtains from $\Phi \in L(p', \infty)$ that

$$(Mf)_\mu^*(\xi) \leq B \int_0^\infty \frac{f_\nu^*(t\xi)}{t^{1/p'}} dt = \frac{B}{\xi^{1/p}} \|f\|_{p,1,\nu}.$$

This is the same as $\|Mf\|_{p,\infty,\mu} \leq B \|f\|_{p,1,\nu}$, or Mf is restricted weak type (p, p) . With this observation it will be easy to obtain the weak type behavior of a maximal operator generalizing the one recently studied by E. M. Stein [9].

THEOREM 6. *Let $\mu > 0$, $\nu \geq 0$ be two Borel measures. Let $1 \leq q \leq p$, and let $M_{pq}f(x) = \sup \|f\chi_Q\|_{p,q,\nu}/\|\chi_Q\|_{p,q,\mu}$, where the sup is extended over all cubes centered at x . Then*

$$\|M_{pq}f\|_{p,\infty,\mu} \leq A \|f\|_{p,q,\nu} \quad \text{or} \quad \mu\{x: M_{pq}f(x) > y\} \leq (C/y^p) \|f\|_{p,q,\nu}^p.$$

PROOF. Note that $\|\chi_Q\|_{p,q,\mu} = \mu(Q)^{1/p}$. It is easy to verify that

$$M_{pq}f(x) = [M_{r,1}f^q(x)]^{1/q}, \quad r = p/q,$$

and thus we need only show that $\mu\{x: M_{r,1}f(x) > y\} \leq (C/y^r) \|f\|_{r,1,\nu}^r$. Note that $\|f\chi_Q\|_{r,1,\nu} \sim \int f\chi_Q\psi_Q d\nu$ for some ψ_Q with $\|\psi_Q\|_{r',\infty,\nu} = 1$ or $\psi_Q^*(t) \leq 1/t^{1/r'}$. Hence, if $\phi_Q = \chi_Q\psi_Q/\mu(Q)^{1/r}$, then $\mu(Q)\phi_Q^*(\mu(Q)t) \leq 1/t^{1/r'}$, from which $\Phi \in L(r', \infty)$.

For the usual Hardy-Littlewood maximal operator, $\Phi \in L(p', \infty)$ actually characterizes the restricted weak type behavior. If (u, v) is a pair of weights,

$$\phi_Q(x) = \frac{1}{|Q|} \frac{\chi_Q(x)}{v(x)}, \quad d\mu = u dx, \quad d\nu = v dx.$$

then $\sup \int f\phi_Q d\nu = \sup |Q|^{-1} \int_Q f dx = Mf(x)$. Let $\Phi(t) = \sup_Q \{\mu(Q)\phi_Q^*(\mu(Q)t)\}$.

THEOREM 7. *Let $1 < p < \infty$. Then $\|Mf\|_{p,\infty,\mu} \leq C \|f\|_{p,1,\nu}$ if and only if $\Phi \in L(p', \infty)$.*

PROOF. We need only show that the norm inequality implies $\Phi \in L(p', \infty)$. We use the technique of [3] and choose $f \geq 0$ with $\|f\|_{p,1,\nu} = 1$ and $\int_Q f = \int f(\chi_Q v^{-1})v \geq C \|\chi_Q v^{-1}\|_{p',\infty,\nu}$. For $x \in Q$ we have $Mf(x) \geq (C/|Q|) \int_Q f$. Our assumption is $\mu\{x: Mf(x) > y\} \leq (C/y^p) \|f\|_{p,1,\nu}^p$ and, hence, we get, with $y = |Q|^{-1} \int_Q f$,

$$\mu(Q) \leq C \|f\|_{p,1,\nu}^p \left(\frac{1}{|Q|} \int_Q f \right)^{-p} \leq \frac{2C |Q|^p}{\|\chi_Q v^{-1}\|_{p',\infty,\nu}^p}.$$

From this it follows that

$$\frac{\mu(Q)}{|Q|} \left(\frac{\chi_Q}{v} \right)_\nu^* (\mu(Q)t) \leq \frac{\mu(Q)^{1/p}}{|Q|} \frac{1}{t^{1/p'}} \left\{ \sup_{\tau>0} \tau^{1/p'} \left(\frac{\chi_Q}{v} \right)_\nu^* (\tau) \right\} \leq C/t^{1/p'},$$

and $\Phi \in L(p', \infty)$.

6. In this section we will present two generalizations of Theorem 1 to abstract measure spaces.

(i) Consider $(X, \mathfrak{N}, \mu, \nu)$ and a measurable map $T: X \rightarrow \mathbf{R}^n$ such that $\mu(T^{-1}(Q)) < \infty$ for every cube $Q \subset \mathbf{R}^n$. Associate with each cube Q a measurable function $\phi_{T^{-1}(Q)}: X \rightarrow [0, \infty)$, with $\text{supp } \phi_{T^{-1}(Q)} \subset T^{-1}(Q)$, and define, for $f: X \rightarrow [0, \infty]$ measurable, the maximal operator

$$\mathfrak{N}f(x) = \sup \int f\phi_{T^{-1}(Q)} d\nu,$$

where the sup is extended over all cubes Q with center $T(x)$.

THEOREM 8.

$$(\mathfrak{N}f)_\mu^*(\xi) \leq A \int_0^\infty \Phi(t) f_\nu^*(t\xi) dt$$

where

$$\Phi(t) = \sup_Q \left\{ \mu(T^{-1}(Q)) (\phi_{T^{-1}(Q)})_\nu^* (\mu(T^{-1}(Q))t) \right\}.$$

For the proof choose for each $x \in E_\tau = \{x: \mathfrak{N}f(x) > \tau\}$, a cube Q_x centered at $T(x)$ for which $\int f \phi_{T^{-1}(Q_x)} d\nu > \tau$. The Besicovitch covering theorem gives us a countable collection $\{Q_j\}$ with $T(E_\tau) \subset \bigcup Q_j$, $\sum \chi_{Q_j} \leq C$. Now proceed exactly as in the proof of Theorem 1, replacing $\mu(Q_j)$ there by $\mu(T^{-1}(Q_j))$.

(ii) We will again consider an abstract measure space $(X, \mathfrak{N}, \lambda)$ and a measurable map $T: X \rightarrow \mathbf{R}^n$ with $\lambda(T^{-1}(Q)) < \infty$, Q cube in $\mathbf{R}^n$. Let $\nu \geq 0$ be a measure on $\mathbf{R}^n$ and associate with each cube $Q \subset \mathbf{R}^n$ a ν -measurable function ϕ_Q with $\text{supp } \phi_Q \subset Q$. For $f: \mathbf{R}^n \rightarrow [0, \infty]$ and $x \in X$ we define

$$M_T f(x) = \sup \int f \phi_Q d\nu,$$

where again the sup is extended over all cubes centered at $T(x)$. As in Theorem 8 one can establish

$$(M_T f)_\lambda^*(\xi) \leq A \int_0^\infty \Phi(t) f_\nu^*(t\xi) dt$$

where

$$\Phi(t) = \sup_Q \left\{ \lambda(T^{-1}(Q)) \phi_{Q,\nu}^* (\lambda(T^{-1}(Q))t) \right\}.$$

As an application we consider $(\mathbf{R}^n, \mu, \nu)$ as in Theorem 1 and assume that $\Phi_0(t) = \sup_Q \{\mu(Q) \phi_{Q,\nu}^*(\mu(Q)t)\}$ is in $L(p', 1)$ for some $1 < p < \infty$. Then we have $\|Mf\|_{p,\mu} \leq A \|f\|_{p,\nu}$, where $Mf(y) = \sup \int f \phi_Q d\nu$, Q centered at y .

THEOREM 9. *If $\lambda(T^{-1}(Q)) \leq C\mu(Q)$, $Q \subset \mathbf{R}^n$, then $\Phi(t) \in L(p', 1)$, and hence $\|M_T f\|_{p,\lambda} \leq A_p \|f\|_{p,\nu}$.*

PROOF. We simply observe that

$$\begin{aligned} \lambda(T^{-1}(Q)) \phi_{Q,\nu}^* (\lambda(T^{-1}(Q))t) &\leq \frac{1}{t} \int_0^{C\mu(Q)t} \phi_{Q,\nu}^*(\tau) d\tau \\ &= \frac{1}{t} \mu(Q) \int_0^{Ct} \phi_{Q,\nu}^*(\mu(Q)\tau) d\tau \leq C \frac{1}{Ct} \int_0^{Ct} \Phi_0(\tau) d\tau \\ &= C\Phi_0^{**}(Ct) \quad [2]. \end{aligned}$$

REMARK. The above hypothesis is a type of Carleson measure condition [1].

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